HIDDEN VARIABLES: ONTOLOGY/EPISTEMOLOGY & CONTEXTUALITY/NON-CLASSICALITY

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Abstract: What does quantum physics tell us about the nature of reality, specifically the parts of reality we do not directly perceive called hidden variables? One may think it could tell us a lot because of our enhanced technological sensing abilities that delve into the realms that quantum physics covers so well. Surprisingly, it seems to surround us in a deeper mystery rather than reveal more of nature's secrets. It seems that we cannot escape from philosophical consideration when dealing with what is hidden in quantum physics. In Part I we will look at how Epistemology and Ontology bear upon Hidden Variables. In Part II we will consider Hidden Variables in the light of Contextuality, and Non-Classicality. Inevitably questions of subjectivity and objectivity arise in dealing with states of observation. How should we think about these states? Perhaps it is a question of the meaning associated with our knowledge of a state—that is, a question of ontology or epistemology. The issue of ontic and epistemic states is particularly important when considering hidden variables in quantum physics because, as one may argue, the interpretation of quantum states as either ontic or epistemic will naturally lead to different assumptions about how reality is constructed; if it is constructed or not. It also raises the question of what attributes we are able to observe simultaneously and that brings contextuality into the discussion. If it turns out that reality is constructed contextually what does that imply about ontological realism? If on the other hand reality is constructed non-contextually what does that imply about ontological realism? Many implications can arise when considering these questions from a quantum physical point of view. In this paper I shall discuss how quantum physics provides some answers to these questions by considering quantum physical states and their measurements.

Keywords: Ontology; Epistemology; Hidden variables; Quantum physics; Contextuality; Reality; Mind

"Reality? What a concept."

Robin Williams

In this paper, I wish to examine several issues regarding just what quantum physics is telling us about the nature of reality, specifically the parts of reality we do not directly perceive called hidden variables. In Part I we will look at how Epistemological and Ontological considerations bear upon the existence of Hidden Variables (HV). In Part II we will consider HV in the light of Contextuality, and Non-Classicality.

We can actuality put both parts under one question: Is there a hidden nature of reality? Some writers in consideration of quantum physics even bring into view a perhaps more fundamental question. They ask us to pose on whether there is an "out there" out there at all. Perhaps, they say, it would be more fruitful to consider the "out there" as a product of the "in here"; in other words, quantum physics is telling us that the universe is a mental construction.[1] Such an answer may appear to be nonsensical until we begin to explore it more carefully from a quantum physical point of view. I shall be doing this in this article.

Now the epistemology and ontology of HV has been examined fairly often in the literature. So let me begin there. Some of what I need to point out here has been described in my earlier paper.[2] Let me repeat some of what I wrote there for clarity and add some new information and some new insights not discussed previously.

PART I: QUANTUM PHYSICS: EPISTEMOLOGICAL OR ONTOLOGICAL HIDDEN VARIABLES?

How should we think about subjective states vs. objective states when considering HV in quantum physics? We would tend to think of subjective states in terms of epistemology—what we know or don't know about what is thought or observed in regard to the "out there" world. For example, I really don't know if you think the world is round or if the moon is made of green cheese. I would in high probability believe you thought "yes" to the first postulate and "no" to the second. My beliefs concerning what you think constitute what we mean by an epistemological state regardless of the actually shape of the earth or the composition of the moon.

Contrarily we tend to think of objective states in terms of ontology as properties of things—real or imagined—out there in the physical world or out there in the minds of others represented as facts. For example, we would classify as an ontological state the fact that the earth is round or the moon is not made of green cheese.

Are such considerations a question of the meaning associated with the word "state"? E.g., is a quantum state to be regarded in the same sense as we regard the classical state of a ball while at rest or while moving after being struck by a bat? What sort of word should we use to describe the batted ball? Shall we say it is in a state of motion—albeit an observed state of motion—and even more—an objective or ontological state? What if I am not watching the ball or am unable to watch it as it moves, but only capable of surmising the trajectory of it based on the sound of the bat hitting the ball. Should I, based on the smack of the bat on the ball, ascribe a probability distribution to the possible trajectories that may have developed while the

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[1] Henry, Richard C. "The Mental Universe." *Nature* 436: 29, 2005. Also see:
Kastrup, Bernardo. "On the Plausibility of Idealism: Refuting Criticisms." *Disputatio* 9 (44): 13-34, 2017.
Kastrup, B. (2014). *Why materialism is baloney: How true skeptics know there is no death and fathom answers to life, the universe, and everything.* Winchester, UK: Iff Books.
[2] F. A. Wolf "Ontology, Epistemology, Consciousness, and Closed Timelike Curves." *Cosmos and History: The Journal of Natural and Social Philosophy, vol. 13, no. 2, 2017*

ball took its course through the air or on the field of the ballpark? If so, what do I call the state of the unseen, yet struck, ball?

I would certainly surmise, believing that there was a baseball hit by a bat, that it did have a trajectory—an objective ontic state of motion—yet not having seen the ball, but only heard the bat strike it, what shall I label the state of the ball under these unseen circumstances? Surely I could and most likely would ascribe a probability distribution to the many possible trajectories such as ascertaining the height of the ball in the air, whether it was foul or fair, how it had top spin or not, etc. Such a probability distribution would be called an epistemic state since my knowledge of the trajectory—that is my knowledge of its ontology—is incomplete.

Here we run into some difficulty dealing with epistemology or epistemic states. Different epistemic states can describe the same ontic state. E.g., the ball could be considered to have a distribution of trajectories and spin possibilities—top spin or back spin—while moving as a fly-ball or as a ground-ball. If the ball had top-spin and was a either a fly- or ground-ball, then both probability distributions, fly or ground, are epistemologically correct descriptions of the baseball's ontological spin because I don't know whether it was a ground or fly ball.

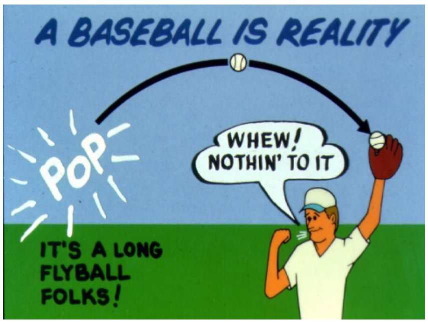

Fig. 1. It's easy if it's ontological.

Or consider what happens when I flip a coin and cover it up before anyone can see the face of the coin showing—heads (H) or tails (T). If the coin was a fair coin, all I can do is assign an epistemological distribution of probabilities, $P_H = ½$, for heads, and, $P_T = ½$, for tails. Such a distribution would constitute the state of the side of the coin showing as an epistemological state. But suppose I peek at the coin, but don't let you see it. Your knowledge of the coin would remain epistemological while mine would suddenly become ontological because I now know the coin is in the ontological state of, e.g., H.

As another classical epistemic example, consider the case of a biased flipping of a coin in one of two distinct ways. In the 1st way the coin has a probability P_1 of coming up H while in the 2nd way the probability for H is $P_2 \neq P_1$. If the coin is flipped and then observed any number of times, regardless of the results obtained, we cannot know for certain by which method the coin was flipped, although the observed frequency of heads resulting could provide a clue provided we knew that the same way of flipping was used for each flip. Not knowing this, the result, H, could have been obtained with either mode of flipping. Hence we cannot uniquely assign either probability P_2 or P_1 and these probabilities remain epistemic although the unobserved method of flipping certainly need not be so.

In another classical epistemic example[3,4,5], consider a die prepared in a special manner that shows the value 2 with a predicted probability of 1/3. We cannot know if the die was prepared in such a way that only prime numbers (2, 3, or 5) were allowed to show (it had these numbers repeated on opposite sides), or if only even numbers (2, 4, or 6) were allowed to show. Each distribution has the number 2 in common, so the distributions are conjoint and epistemic.

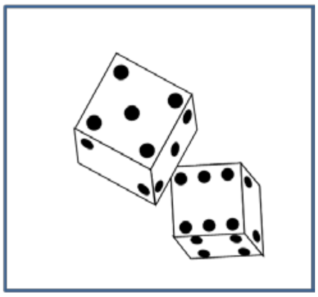

Fig. 2. Loaded dice, but which way?

The implication here is that whenever there are two or more possible ways of preparation or as we shall put it, distributions of HVs that give the same observational result we say these HVs make up an epistemological conjoint state. While this may

[3] M. F. Pusey, J. Barrett, and T. Rudolph. "On the reality of the quantum state." http://arxiv.org/abs/1111.3328v2. Also see: *Nature Physics* (2012), published online 06 May 2012.
[4] M. F. Pusey, J. Barrett, and T. Rudolph. "The quantum state cannot be interpreted statistically." http://arxiv.org/abs/1111.3328v1.
[5] E. S. Reich. "A boost for quantum reality." *Nature* Vol. 485. 10 May 2012. pp 157-8. This example was given to Reich by Terry Rudolph.

seem obvious in classical consideration (i.e., non-quantum physical) there are other possible implications when quantum physics is brought to bear. In quantum physics we need to carefully reconsider ontology and epistemology and in so doing a lot of confusion can arise.

In a remarkable remark, physicist E. T. Jaynes once stated:[6]

> We believe that to achieve a rational picture of the world it is necessary to set up another clear division of labor within theoretical physics; it is the job of the laws of physics to describe physical causation at the level of ontology, and the job of probability theory to describe human inferences at the level of epistemology. The Copenhagen interpretation scrambles these very different functions into a nasty omelet in which the distinction between reality and our knowledge of reality is lost.

Suppose we prepare this omelet by giving it a different stir. Is the quantum wave function (QWF) epistemologically imagined or ontologically real? In an earlier *Nature* review[7] E. S. Reich discussed the work of three physicists: M. F. Pusey, J. Barrett, and T. Rudolph (PBR).[8] PBR, basing their work on a number of previous epistemic vs. ontic considerations dating all the way back to the Einstein-Bohr debate at the 1927 Solvay conference in Brussels and continuing with the 20th and 21st century work of many others notably Einstein, Ballentine, Bohm, Bell, Peierls, Caves, Fuchs, Harrigan and Spekkens, Kochen and Specker (about whom I have more to say later), and others,[9] once again throw down the gauntlet of uncertainty by attempting to provide an

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6 E. T. Jaynes. "Clearing up mysteries." In the Proceedings Volume: *Maximum entropy and Bayesian methods.* J. Skilling (ed.). Kluwer Academic Publ. Dordrecht, Holland (1989). pp 1-27.

7 E. S. Reich. *Ibid.*.

8 M. F. Pusey, J. Barrett, and T. Rudolph. "On the reality of the quantum state." http://arxiv.org/abs/1111.3328v2. Also see: *Nature Physics* (2012) doi:10.1038/nphys2309. Received 05 March 2012 Accepted 11 April 2012, Published online 06 May 2012.

9 Einstein, A., Podolsky, B. & Rosen, N. "Can quantum-mechanical description of physical reality be considered complete?" *Phys. Rev.* 47, 777_780 (1935). See also:

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Bell, J. (1964). "On the Einstein Podolsky Rosen paradox." *Physics*, 1 (3), 195-200.

Peierls, R. E. *Surprises in Theoretical Physics 32* (Princeton Univ. Press, 1979).

Caves, C. M., Fuchs, C. A. & Schack, R. "Quantum probabilities as Bayesian probabilities." *Phys. Rev. A* 65, 022305 (2002).

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ontic real view of the QWF, something that even Bohr most likely was not ever considering. Jaynes even pointed out that the famous Bohr-Einstein debate was actually never resolved in favor of Bohr at Solvay in 1927—although common thinking even among physicists is that it was—when you consider that the two physicists were not discussing the same physics. Bohr was only thinking about epistemic physics while Einstein was considering only ontic physics. Hence while Bohr believed quantum physics was certainly epistemically complete (like classical thermodynamics with hidden variables like gaseous atoms bustling around), Einstein was equally correct in believing that quantum physics wasn't ontologically complete (like Newtonian mechanics, perhaps thinking there were also hidden variables like atoms bustling around which are not taken into account).

The conflict between all we know about the physics of quantum systems and what we say or believe is real about them is brought forward dramatically with the concept of the QWF. Is the QWF ontic or merely epistemic? To decide on the ontology or epistemology of a QWF, the old argument, known as the HV theory dating back to the

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Montina, A. "State-space dimensionality in short-memory hidden-variable theories." *Phys. Rev. A* 83, 032107 (2011).

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Fuchs, C. A. "QBism, the perimeter of quantum Bayesianism." Preprint at http://arxiv.org/abs/1003.5209 (2010.

mid 20th century, is revisited. This theory was probably most emphasized by David Bohm (who formulated from standard quantum physics an ontic QWF that influenced a real particle). Later it was revisited by Bell, in his famous no-go theorem involving a QWF describing two quantum-entangled separated particles ala Bohm's version of the Einstein, Podolsky, and Rosen (BEPR) paradox. BEPR showed that such a QWF could not be local (measurements made on one particle at one spacetime location could influence and change the QWF and therefore the outcome of measurement on the other particle at a distant (spacelike) spacetime location simultaneously). Bell's theorem shows that any hidden variable theory must involve nonlocal influences at the ontic level, regardless of what you think of the QWF. Hence one might conclude from Bell's famous HV theorem (ala Einstein) that QWFs are epistemological rather than ontological since two observers could have different beliefs about the quantum state of their respective spacelike separated particles.[10]

Quantum physical HV theories all have one thing in common; they all have ontic definite-valued hidden states underlying the QWF which also underlie classical physics and thermodynamics. A specification of these HVs should reveal the results of a measurement of any property or observable.[11] So the question is what would one need to do to a HV theory to make the QWF ontological? This is precisely what PBR attempt by making a particular assumption: If a specification of a HV uniquely determines a QWF, then the QWF is ontic. If, on the other hand, specification of a HV does not uniquely determine a QWF, the QWF is said to be epistemic. Of course such an ansatz may not be sufficient to prove ontology of quantum wave functions. It just has us consider the question of ontology of QWFs when such a restriction is in place.

Classical physics epistemics

Let me now give you a simple example of the difference between ontic and epistemic reality taken from classical physics. Consider a ball with mass, $m = ½$, attached to a spring with spring constant, $k = 2$. (See Fig. 3.)

[10] Indeed Einstein did make this conclusion based on the EPR argument. However, it is not a conclusion of Bell's theorem and certainly not Einstein's conclusion based on Bell's work because he was dead at the time. In fact, Bell's theorem rather stymies this line of argument, since it says that you will still have nonlocal influences even if the wavefunction is epistemic, so this move does not solve the problem of nonlocality.

[11] One may need to allow for the fact that measurements might be fundamentally noisy or stochastic and only demand that HVs specify probabilities for any measurement outcome.

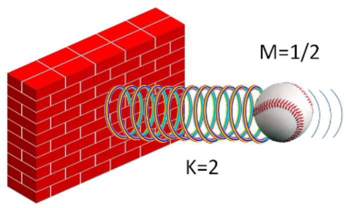

Fig. 3. Ball and spring attached to a wall.

Such a system is known as a simple harmonic oscillator (SHO)—stretch or compress the spring and the SHO "springs" into motion with the ball having momentum, p, and a position, x, relative to its unstretched or uncompressed 0 position, and constant energy, $E = p^2 + x^2$. I'll use a single variable λ to denote the ontic pair (p, x). Suppose that someone unknown to us stretches the spring an unknown initial distance, x_o, within a range $1 \leq x_o \leq 2$ or in a 2nd range $3 \leq x_o \leq 4$. If you think of a two dimensional space with orthogonal coordinate axes, p and x, the above energy equation describes a circle contained within one of the two sets of concentric thickened rings centered about the coordinate origin. Such a space is a simple example of what is called a phase space which in general has n dimensions of ps and xs. Each point on a ring provides a momentum and position of the ball which, even if not observed, hence hidden, are ontic variables. At no time do the different rings have common points of overlap.

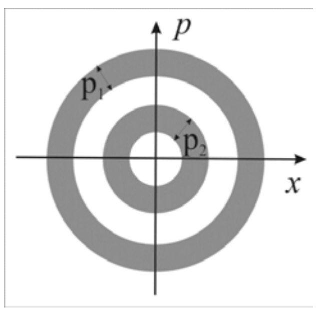

Fig. 4. Disjoint epistemic probability distributions in phase space for a SHO (see text.)

We can think of the rings as disjoint probability distributions, $p_1(\lambda)$ and $p_2(\lambda)$, of positions and momenta—disjoint because we never have any λs in common—the rings are concentrically nested (See Fig. 4.). Each λ may be a uniformly distributed (over time) HV satisfying the SHO energy equation. However, as I said, these simple distributions would be disjointed. Hence $p_1(\lambda)p_2(\lambda) = 0$ always since each λ uniquely determines its own distribution (in which ring it belongs). Consequently if there was a state α_1 associated with $p_1(\lambda)$ and a state α_2 associated with $p_2(\lambda)$, then specification of the value of λ would uniquely determine which state, α_1 or α_2, we would be in. We could, although it is clearly not necessary, view the λs as HVs and declare the states as ontic since each λ determines a unique α.

Suppose we now reconsider the initial preparation of the SHO. At $t = 0$, that unknown someone simply decides to stretch the spring a certain distance, x_o, an amount in the range, $1 \leq x_o \leq 3$, and lets it go [12]. We would then find a thick ring-band of different energy possibilities in the phase plane. Or if another unknown person prepares the SHO in the range, $2 \leq x_o \leq 4$, and lets it go, we would then find a 2nd thick ring-band of possibilities. The two circular bands now form overlapping concentrically nested distributions (see Fig. 5.). Now we have the two distributions, $p_1(\lambda)$ and $p_2(\lambda)$ overlapping. Then $p_1(\lambda)p_2(\lambda)\neq 0$ in the overlapping area $2 \leq x_o \leq 3$ and each λ no longer uniquely determines its own α state. A specification of λ in the overlapping probability distribution would indicate we were in either the α_1 or α_2 state and that would make the states epistemic.

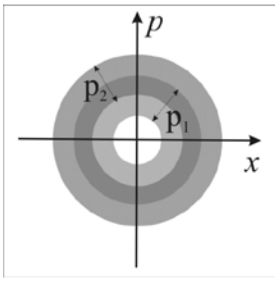

Fig. 5. Conjoint (overlapping dark grey) epistemic probability distributions in phase space for a SHO (see text.)

PBR's proof is based on a contradiction that arises between the probability

12 In this SHO example (with m=½ and k=2), assuming at t=0, the spring is stretched to a distance, $\sqrt{E}$, we get $x=(\sqrt{E})\cos(2t)$ and $p=(-\sqrt{E})\sin(2t)$. The point in the phase plane rotates clockwise around the circle completing the cycle in the period of π. The probability density is simply a constant, $dP/dt=1/\pi$, for all such circles regardless of the energy. Indeed that's why spring clocks work.

predictions of quantum physics when QWFs are considered to be "ontological" (their respective HV probability distributions are disjoint) and the same predictions based on "epistemic" QWFs (their respective HV probability distributions are conjoint). They consider this contradiction in a series of ever increasingly complex arguments that includes a calculation eventually involving *n* identically prepared and uncorrelated independent states as well as noise considerations. Accordingly, whenever QWFs of observables are governed by disjoint distributions of ontic HVs, these QWFs are uniquely determined and must be ontic even though their respective distributions are epistemic (similar to arguments made in statistical mechanics). Thus if the states of a quantum system are specified by QWFs which are determined by disjoint epistemic distributions over ontic variables, the QWFs are as ontic or real as any observable in physics. On the other hand, if such distributions governing these QWFs are conjoint, that is, they have values of ontic HVs in common; the QWFs are epistemic or merely represent knowledge (probabilities) of observables in question.

Simple quantum physics ontology and epistemology

Before we look at PBR's argument, I want to explain a little more about why overlapping probability distributions lead to a contradiction in the quantum physical predictions. Consider for simplicity a top hat probability distribution, $p_\psi(\lambda)$. We shall be looking at two special cases ψ = N and ψ = S, (you can think of these states as polar opposites) associated with orthogonal QWFs, N and S, respectively (that is, $\langle N|S\rangle = 0$), which have a common overlapping area of an HV, λ (λ could also indicate a set of HVs). A common λ means simply that both $p_S(\lambda) \neq 0$, and $p_N(\lambda) \neq 0$ in the overlap as shown in Fig. 6.

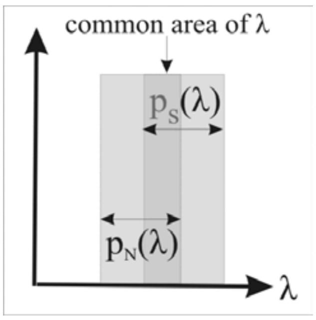

Fig. 6. Conjoint top hat (overlapping) epistemic probability distributions for orthogonal quantum physics states.

1st case: Now consider the probability of obtaining a measurement of N and suppose that this probability depends only on the HV λ. We can write it as a conditional (Bayesian) measurement probability, $M(N|\lambda)$. To obtain the total probability, $P(N|\psi)$, that is, to get the probability for result N for any QWF, ψ, we must calculate $P(N|\psi) = \int M(N|\lambda)p_\psi(\lambda)d\lambda$. That is, we multiply the probability of obtaining a result for a given λ by the distribution function, $p_\psi(\lambda)$, specific to the chosen QWF, ψ, and integrate over all λ. From the Born Rule of quantum physics, $P(N|\psi) = \langle\psi|N\rangle\langle N|\psi\rangle$.

2nd case: Next consider a measurement of S which is also given by a conditional measurement probability, $M(S|\lambda)$, which is also clearly dependent only on HV, λ. Now suppose we wish to obtain the probability of getting the result, S. Similarly, to obtain the total probability $P(S|\psi)$ for getting the result S, we must have $P(S|\psi) = \int M(S|\lambda)p_\psi(\lambda)d\lambda$. And again from the Born Rule: $P(S|\psi) = \langle\psi|S\rangle\langle S|\psi\rangle$.

Now if $M(S|\lambda)$ and $M(N|\lambda)$ are the only probabilities of obtaining values by measurements, and since there are only two such values possible, then clearly $M(S|\lambda) + M(N|\lambda) = 1$. There can be no other result possible and this must hold for every λ value. In plain language, specifying λ must lead to unity probability when all possible results of a measurement are taken into account with ontic variable λ specified. For example, λ could be a simple option, λ_q or λ_d, for an unseen biased coin—use a quarter or use a dime. Using a quarter suppose $M(H|\lambda_q) = .25$ and $M(T|\lambda_q) = .75$ or using a dime suppose $M(H|\lambda_d) = .65$ and $M(T|\lambda_d) = .35$. In each HV option, dependent on the value of λ, head (H) and tail (T) are clearly orthogonal results after a toss of the coin. Again, as in the other coin example, after many such observations we could only guess the HV of the coin was a dime or a quarter because of the relative frequencies of heads to tails appearing provided we knew that the same type of coin was used each time. Otherwise we would never know which coin was used.

However, as simple as is this N or S case, it leads to a contradiction with the Born rule of quantum physics that arises when you put $\psi = S$ in the 1st case, and $\psi = N$ in the 2nd case. Since S and N are orthogonal (they both cannot occur), $\langle S|N\rangle = 0$. Hence in the 1st case we get,
$\langle\psi|N\rangle\langle N|\psi\rangle = \langle S|N\rangle\langle N|S\rangle = P(N|S) = \int M(N|\lambda)p_S(\lambda)d\lambda = 0$, and in the 2nd case, $\langle\psi|S\rangle\langle S|\psi\rangle = \langle N|S\rangle\langle S|N\rangle = P(S|N) = \int M(S|\lambda)p_N(\lambda)d\lambda = 0$. If these integrals are to be zero, then the integrands have to be zero for every value of λ because both $M(N|\lambda)$ and $M(S|\lambda)$ as well as $p_S(\lambda)$ and $p_N(\lambda)$ are positive functions. Therefore, in particular, these integrands have to be zero in the overlapping region. But given that both $p_S(\lambda) \neq 0$ and $p_N(\lambda) \neq 0$ in the overlapping region, that is, we have

overlapping distributions in λ space (see Fig. 6.), these results can only occur if both $M(N|\lambda) = 0$ and $M(S|\lambda) = 0$ which contradicts $M(S|\lambda) + M(N|\lambda) = 1$.

Hence for this simple orthonormal case, we cannot have both $p_S(\lambda)$ and $p_N(\lambda)$ possessing nonzero values for any common λ. In brief they cannot have overlapping hidden variables. This means that a specification of λ leads to a unique ψ, either S or N (as in the quarter/dime example above), and we can therefore take it that for any common λ, $p_S(\lambda)p_N(\lambda) = 0$, so in both cases either $p_S(\lambda)$ or $p_N(\lambda)$ must be zero. PBR might call this a necessary step to proving that a QWF is an ontological function, but this proof only includes orthogonal QWFs, |N> and |S> as indicated in Fig. 7. To be both necessary and sufficient one would need to show that the probability distribution $p_N(\lambda)$ for |N> and any other probability distribution $p_\psi(\lambda)$ for a QWF |ψ> cannot have any overlap even if $\langle N|\psi\rangle \neq 0$.

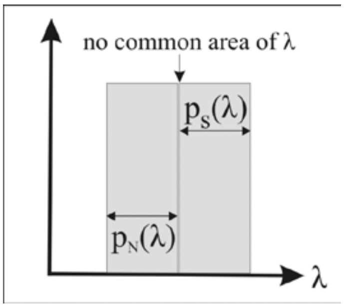

Fig. 7. Disjoint epistemic probability distributions for orthogonal quantum physics states leading to ontic states |N> and |S>.

More complex quantum physics ontology and epistemology

In the above case we only considered orthogonal QWFs, N and S, and found them to be ontic according to PBR's supposition. Can we make the argument that ψ is real in any case including nonorthogonal situations? To fully answer this query, we would need to look at the case when possible quantum states, α and β, are not orthogonal. One might think that since two such QWFs, |α> and |β> do overlap, i.e., $\langle\beta|\alpha\rangle \neq 0$, one might find no contradiction in having both $p_\alpha(\lambda) \neq 0$, and $p_\beta(\lambda) \neq 0$. Hence both α and β could be epistemic and still satisfy the Born rule of quantum physics.

PBR dispel that possibility by first considering nonorthogonal states of the same simple system as above that is prepared with compass directions |N> or |E>, where $|E\rangle = (|N\rangle + |S\rangle)/\sqrt{2}$, $|W\rangle = (|N\rangle - |S\rangle)/\sqrt{2}$. Here we have

$\langle N|S\rangle = \langle E|W\rangle = 0$, respectively orthogonal, but $\langle N|E\rangle = 1/\sqrt{2}$, hence N and E are not orthogonal.[13] We shall again assume that the QWF, $|\psi\rangle$, (either $|N\rangle$ or $|E\rangle$) is dependent on a HV distribution $p_\psi(\lambda)$ similar to what we did in the orthogonal case above. One can recognize these "directional" states as *spinors*, i.e., spin-½ states, wherein $|N\rangle$ means spin up in the z direction, $|S\rangle$ means spin down in the z direction, $|E\rangle$ means spin up in the x direction, and $|W\rangle$ means spin down in the x direction.

The system is to be prepared in one of two ways such that one preparation produces $|N\rangle$ with unity probability, $P(N|N) = \int M(N|\lambda)p_N(\lambda)d\lambda=1$, arising from an epistemic $p_N(\lambda)$ distribution, while a second kind of preparation produces $|E\rangle$ with unity probability, $P(E|E) = \int M(E|\lambda)p_E(\lambda)d\lambda=1$, arising from epistemic distribution, $p_E(\lambda)$. The aim: <u>If</u> a specification of λ yields a specific QWF, $|\psi\rangle$, orthogonal or not to any other QWF, $|\alpha\rangle$, <u>then</u> $|\psi\rangle$ must be ontic and therefore an objective real "thing" "out there" independent of any observer. So, accordingly, in the case involving states, $|N\rangle$ and $|E\rangle$, in spite of the nonorthogonality of these states, the two distributions, $p_N(\lambda)$ and $p_E(\lambda)$ must be disjoint, $p_N(\lambda)p_E(\lambda)=0$, as shown in Fig. 7 only substitute E for S.[14]

On the other hand, if λ lies within a region where $|N\rangle$ and $|E\rangle$ have conjoint distributions, i.e., $p_N(\lambda)$ and $p_E(\lambda)$ overlap so that $p_N(\lambda)p_E(\lambda) \neq 0$, then $|\psi\rangle$ cannot be ontic and must be epistemic as shown in Fig. 6 (again substitute E for S).[15] In brief, an epistemic $|\psi\rangle$ results in a contradiction with the prediction of quantum physics just as we saw in the above N and S orthogonal case.

To clarify their argument, I will follow PBR with a slight change of notation. PBR have us consider a quantum physical situation in which two such identical, but separate, preparations $|\psi_1\rangle$ and $|\psi_2\rangle$ are independently made using HVs, λ_1 and λ_2, wherein both HVs lie within identical HV spaces; we have essentially two copies of the same hidden variable space. Consequently these preparations result in the uncorrelated joint quantum state $|\psi_1\rangle|\psi_2\rangle$, since they are produced from independent HVs. It is important to realize that PBR assume that both λ_1 and λ_2 lie within corresponding, respectively, identical but independent HV spaces. Thus each separate space of HVs

[13] This sounds peculiar since clearly the directions are perpendicular. However perpendicular in space does not necessarily mean the same thing as orthogonal in quantum physics. For those who know a little quantum physics; two quantum states α and β are orthogonal if and only if $\langle\alpha|\beta\rangle = 0$.

[14] That is, there is no overlap of these probability distributions, so $p_N(\lambda)\,p_E(\lambda) = 0$. So this means either $p_N(\lambda) = 0$ or $p_E(\lambda) = 0$ for all λ.

[15] Here there is an overlap, so $p_N(\lambda)\,p_E(\lambda) \neq 0$. So that means both $p_N(\lambda) \neq 0$ and $p_E(\lambda) \neq 0$ for λ within the overlap region.

contains an identical range, $\rho \geq 0$, over which probability distributions are conjoint. Consequently each preparation produces its own corresponding HV, λ_i, resulting in identical overlapping probability distributions of |N> and |E>, wherein, $p_N(\lambda_1)p_E(\lambda_1) \neq 0$ and $p_N(\lambda_2)p_E(\lambda_2) \neq 0$, provided λ_1 lies within the overlapping range, ρ, and λ_2 lies within the same correspondingly identical overlapping range, ρ, as shown in Fig. 8.

That is, both systems are prepared in such a manner that we cannot uniquely determine |N> or |E>. PBR also assume the probability distribution functions, $p_N(\lambda_i)$ and $p_E(\lambda_i)$, are the same for i=1 or 2. Since these are independent preparations, both $p_{\psi 1}(\lambda_1) \neq 0$ and $p_{\psi 2}(\lambda_2) \neq 0$ whenever λ_1 and λ_2 are each found in the same range, ρ. In Fig. 8 we are essentially duplicating the scenario shown in Fig. 6 for each copy.

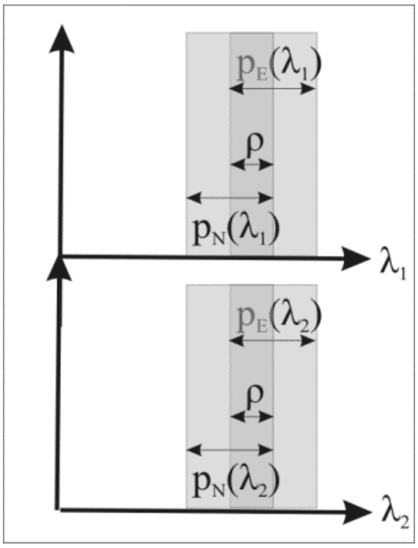

Fig. 8. Conjoint top hat (overlapping) epistemic probability distributions for two identical systems with non-orthogonal quantum physics states.

So after preparing the joint system with both λ_1 and λ_2 in their corresponding conjoint ρ ranges, we obtain the following epistemic (possible) results for $|\psi_1\rangle|\psi_2\rangle$: |N>|N>, |N>|E>, |E>|N> or |E>|E>. All we need now is to specify the basis for making a measurement of the joint system. Suppose now that the two systems are brought together and measured using (projected onto) the following orthonormal entangled base states:

$|1\rangle = (|N\rangle|S\rangle + |S\rangle|N\rangle)/\sqrt{2}$,

$|2\rangle = (|N\rangle|W\rangle + |S\rangle|E\rangle)/\sqrt{2}$,

$|3\rangle = (|E\rangle|S\rangle + |W\rangle|N\rangle)/\sqrt{2}$, and,

$|4\rangle = (|E\rangle|W\rangle + |W\rangle|E\rangle)/\sqrt{2}$. eqns. 01.

These four states are maximally entangled and orthogonal ($\langle i|j\rangle = 0$, unless $i = j$, and then $\langle i|i\rangle = 1$.) Consequently the probability for obtaining a result, i, $P(i|\psi_1\psi_2)$, given that the joint wave function, $|\psi_1\psi_2\rangle = |\psi_1\rangle|\psi_2\rangle$, can be expressed in a similar manner as for the simple case above. Following the above example and the Born rule, we have for the joint probability,
$P(i|\psi_1\psi_2) = \langle\psi_1\psi_2|i\rangle\langle i|\psi_1\psi_2\rangle = \iint M(i|\lambda_1,\lambda_2)p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2)d\lambda_1 d\lambda_2$,
where the probability of obtaining a joint measurement, M, of state $|i\rangle$ now depends on two HVs, λ_1 and λ_2 and we write it accordingly as a conditional (Bayesian) probability, $M(i|\lambda_1,\lambda_2)$. Consequently, we cover all of our four bases and find for any chosen pair of HVs, λ_1 and λ_2, $M(1|\lambda_1,\lambda_2) + M(2|\lambda_1,\lambda_2) + M(3|\lambda_1,\lambda_2) + M(4|\lambda_1,\lambda_2) = 1$. This says that the probabilities of obtaining a result for i, $1 \le i \le 4$, now depends on both given λ_1 and λ_2 values. Change those values and the individual $M(i|\lambda_1,\lambda_2)$ may change, as in the case of the quarter and dime; but they will always sum to unity regardless whether or not the chosen values of λ_1 and λ_2 fall within the ranges of $\rho \ge 0$.

The question is: what are the probabilities of the results of measurement using (projecting onto) these entangled base states according to the Born rule of quantum physics? It isn't too difficult to see that there are four cases in which we get predictions of zero probabilities—the result of a measurement will be to <u>not</u> find a specific result.

As we see next this fact leads to a contradiction if λ_1 and λ_2 fall within the overlapping ranges of ρ, thus producing non-vanishing conjoint probability distributions. It is here where the independence and conjointness of the two individually overlapping probability distributions, $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) \neq 0$, play their roles.

In the first case, consider $P(1|NN) = \langle NN|1\rangle\langle 1|NN\rangle = 0$ as can be seen by inspection of eqns. 01. Therefore, $\iint M(1|\lambda_1,\lambda_2)p_N(\lambda_1)p_N(\lambda_2)d\lambda_1 d\lambda_2$ must be 0. But since λ_1 and λ_2 have non-vanishing probability distributions, $p_N(\lambda_1)p_N(\lambda_2) \neq 0$, it follows that $M(1|\lambda_1,\lambda_2) = 0$. A similar line of reasoning applies to $P(2|NE) = \langle NE|2\rangle\langle 2|NE\rangle = 0$, where $p_N(\lambda_1)p_E(\lambda_2) \neq 0$, and for $P(3|EN) = \langle EN|3\rangle\langle 3|EN\rangle = 0$, where $p_E(\lambda_1)p_N(\lambda_2) \neq 0$, and finally for $P(4|EE) = \langle EE|4\rangle\langle 4|EE\rangle = 0$, where $p_E(\lambda_1)p_E(\lambda_2) \neq 0$. Remember we are assuming that $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) \neq 0$, corresponding to λ_1 and λ_2 each falling within the range of ρ and these are the only cases of concern.

Therefore we would conclude for these particular values of λ_1 and λ_2, within the ranges of ρ where $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) \neq 0$, in each of the vanishing probabilities, $P(i|\psi_1\psi_2) = 0$, we must have $M(1|\lambda_1,\lambda_2) = 0$, $M(2|\lambda_1,\lambda_2) = 0$, $M(3|\lambda_1,\lambda_2) = 0$, and $M(4|\lambda_1,\lambda_2) = 0$, which contradicts the equation: $M(1|\lambda_1,\lambda_2) + M(2|\lambda_1,\lambda_2) + M(3|\lambda_1,\lambda_2) + M(4|\lambda_1,\lambda_2) = 1$, which is valid for all values of λ_1 and λ_2. The only way out of the contradiction is, of course, to deny the non-vanishing overlapping probability distributions, where λ_1 and λ_2 are within the supported "overlapping" ranges of values of ρ, $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) \neq 0$, can ever occur. Thus $P(1|NN) = 0$ implies that $p_N(\lambda_1)p_N(\lambda_2) = 0$, $P(2|NE) = 0$ implies that $p_N(\lambda_1)p_E(\lambda_2) = 0$, $P(3|EN) = 0$ implies that $p_E(\lambda_1)p_N(\lambda_2) = 0$, and $P(4|EE) = 0$ implies that $p_E(\lambda_1)p_E(\lambda_2) = 0$. In each case it's necessary and sufficient that only one of the pairs of $p_{\psi i}(\lambda_i)$s need vanish to rule out any overlap and thus rule in that all such ψ_is are ontological. Having either $p_{\psi i}(\lambda_i)$ vanish means $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) = 0$, and consequently since both ψ_1 and ψ_2 are either N or E then the condition $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) \neq 0$ is equally ruled out for each ψ_i. Thus for any pair of nonorthogonal ψ_is, the Born rule of quantum physics cannot be satisfied, if their respective HV probabilities overlap.

Simple illustration of the BPR theorem for two states

Of course, it could be that for most values of λ_1 and λ_2, outside the range of ρ, or indeed if $\rho = 0$, the condition $p_{\psi 1}(\lambda_1)p_{\psi 2}(\lambda_2) = 0$ need not arise to have $P(i|\psi_1\psi_2)=0$ and for these cases no contradiction arises. To further clarify the argument consider Fig. 9, where I show a possible set of conditional measurement probability distributions, $M(i|\lambda_1,\lambda_2)$, consistent with nonoverlapping top hat probability distributions shown in Fig. 8 with $\rho = 0$. Each conditional measurement probability distribution consists of a quilt of four patches with $M(i|\lambda_1,\lambda_2)$ being constant in each patch and $i \in (1,4)$. The darkest patch has $M(i|\lambda_1,\lambda_2) = 0$, the light grey patches have $M(i|\lambda_1,\lambda_2) = .25$, and the nearly white patch has $M(i|\lambda_1,\lambda_2) = .50$. One can see by inspection that $M(1|\lambda_1,\lambda_2) + M(2|\lambda_1,\lambda_2) + M(3|\lambda_1,\lambda_2) + M(4|\lambda_1,\lambda_2) = 1$ for any pair of values, (λ_1,λ_2), in the quilt. So long as $\rho = 0$, we never see any contradiction arising with the Born Rule because the disjoint probability distributions, $p_{\psi 1}(\lambda_1)$and $p_{\psi 2}(\lambda_2)$, are consistently defined within the same boundaries as the quilted measurement probabilities, $M(i|\lambda_1,\lambda_2)$. It is only when $p_{\psi 1}(\lambda_1)$ and $p_{\psi 2}(\lambda_2)$ exceed those quilted boundaries that contradictions arise as indicated next.

If we have $\rho > 0$, then these measurement probabilities, $M(i|\lambda_1,\lambda_2)$, lead to contradiction with the Born rule. To see this in each of the four cases, let us again consider our conjoint top hat probability distributions, as shown in Fig. 7 such that,

$p_N(\lambda_1) = p_N(\lambda_2) = 1/(1 + \rho/2)$ in the ρ-extended range, when $0 \leq \lambda_1 \leq (1 + \rho/2)$ and $0 \leq \lambda_2 \leq (1 + \rho/2)$, resp., and 0 elsewhere. And similarly for $p_E(\lambda_1) = p_E(\lambda_2) = 1/(1 + \rho/2)$ in the ρ-extended ranges, $(1 - \rho/2) \leq \lambda_1 \leq 2$ and $(1-\rho/2) \leq \lambda_2 \leq 2$, resp., and 0 elsewhere. Consequently we have the normalized probabilities, $\int p_N(\lambda_i)d\lambda_i = \int p_E(\lambda_i)d\lambda_i = 1$, for $i = 1, 2$.

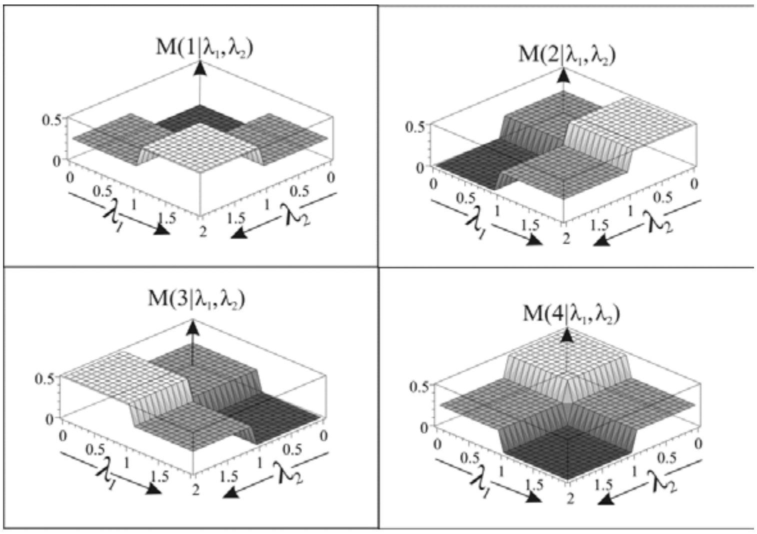

Fig. 9. Three dimensional views of quilted, stepped, conditional measurement probabilities, $M(i|\lambda_1,\lambda_2)$, consistent with disjoint top hat probability distributions for two identical systems with non-orthogonal quantum physics states.

Case 1. Let us now examine the first case where $P(1|NN) = \langle NN|1\rangle\langle 1|NN\rangle = \iint M(1|\lambda_1,\lambda_2)p_N(\lambda_1)p_N(\lambda_2)d\lambda_1 d\lambda_2 = 0$, according to the Born Rule. There is no problem for $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$; we simply have on this patch of the λ-quilt, $M(1|\lambda_1,\lambda_2) = 0$. However in the overlapping ranges, $1 < \lambda_1 \leq (1 + \rho/2)$ and $1 < \lambda_2 \leq (1 + \rho/2)$, $M(1|\lambda_1,\lambda_2) = .5$, and consequently $P(1|NN) = \rho^2/[8(1 + \rho/2)^2] \neq 0$, in contradiction of the Born Rule.

Cases 2, 3, and 4. A similar line of reasoning applies for the other cases: $P(2|NE) = \langle NE|2\rangle\langle 2|NE\rangle = \iint M(2|\lambda_1,\lambda_2)p_N(\lambda_1)p_E(\lambda_2)d\lambda_1 d\lambda_2 = 0$, $P(3|EN) = \langle EN|3\rangle\langle 3|EN\rangle = \iint M(3|\lambda_1,\lambda_2)p_E(\lambda_1)p_N(\lambda_2)d\lambda_1 d\lambda_2 = 0$, and $P(4|EE) = \langle EE|4\rangle\langle 4|EE\rangle = \iint M(4|\lambda_1,\lambda_2)p_E(\lambda_1)p_E(\lambda_2)d\lambda_1 d\lambda_2 = 0$, according to the Born Rule.

Of course, in each case, in the limit where $\rho \to 0$, no contradiction arises and the correct results for the measurement probabilities are obtained. Thus, e.g., from the top right hand corner of Fig. 8 dealing with measurements projected onto the |2> state we find: $P(2|NE) = \langle NE|2\rangle\langle 2|NE\rangle = \iint M(2|\lambda_1,\lambda_2)p_N(\lambda_1)p_E(\lambda_2)d\lambda_1 d\lambda_2 = 0$, $P(2|NN) = \langle NN|2\rangle\langle 2|NN\rangle = \iint M(2|\lambda_1,\lambda_2)p_N(\lambda_1)p_N(\lambda_2)d\lambda_1 d\lambda_2 = .25$, $P(2|EN) = \langle EN|2\rangle\langle 2|EN\rangle = \iint M(2|\lambda_1,\lambda_2)p_E(\lambda_1)p_N(\lambda_2)d\lambda_1 d\lambda_2 = .50$, and $P(2|EE) = \langle EE|2\rangle\langle 2|EE\rangle = \iint M(2|\lambda_1,\lambda_2)p_E(\lambda_1)p_E(\lambda_2)d\lambda_1 d\lambda_2 = .25$, all consistent with the Born Rule leading to unity probability when summed. Similar results follow for all the other measurements projected onto the |i> state, with i = 1, 3, and 4.

DISCUSSION OF PART I: DISJOINT HV→REAL QWFS

To prove or disprove whether or not any general QWF, $|\alpha\rangle$ is ontic is quite an accomplishment even for a limited HV, but clever, approach as taken by PBR. To establish that a given $|\alpha\rangle$ is ontic, you have to construct an argument showing that for any other QWF, $|\beta\rangle$, even when $\langle\beta|\alpha\rangle \neq 0$, it is always possible to find such a contradiction as shown above. PBR use *n* identically prepared and uncorrelated independent QWFs (I looked at *n*=2) generating a QWF, $|\Psi\rangle = |\psi_1\rangle|\psi_2\rangle \ldots |\psi_n\rangle$, where each QWF is either $|\alpha\rangle$ or $|\beta\rangle$. $|\Psi\rangle$ is projected onto an entangled QWF measuring device (a combination of various gates and other devices used in quantum computers called a measurement circuit) that jointly measures the *n* systems in such a manner that there is always at least one of the 2^n QWFs predicted with zero probability. Indeed this is a very clever idea as one can nearly always show[16] that $|\Psi\rangle$, being a product of independent QWFs, must consist of independent ontic states.

On the other hand if a measurement of a state with zero probability ever occurs (e.g., corresponding to an EN measurement when a not-EN state was prepared, indicating a violation of the predicted quantum probabilities, does that indicate Einstein was right after all and quantum physics is ontologically incomplete?[17]

Could this be proven experimentally? All one would need to do is show that the condition of never finding a zero probability case in any the 2^n possible cases would possibly do it. Suppose that indeed one were to find all (measurement) projections onto such entangled base states devices never occurring with zero probability.[18] According to

[16] PBR also carry out an error analysis to complete their proof.

[17] Such a violation would tell us that it is possible, i.e., not in conflict with experimental results, that the wavefunction is epistemic.

[18] Matt Leifer in an email to me pointed out that from any epistemic HV theory, you can always construct one that is ontological and gives exactly the same predictions. Such an argument is given in M. Schlosshauer and A. Fine, "Implications of the Pusey-Barrett-Rudolph no-go theorem."

PBR the epistemic nature of QWFs in violation of quantum physics, would be established. Einstein would emerge victorious and we would need a new physics beyond quantum physics.

In summary we have a logical proof here: For two or more QWFs the Born rule (TBR) implies disjoint HV probability distributions (DPD), TBR→DPD. However DPD does not necessarily imply the Born rule, ~(DPD→TBR). They are not equivalent. The important statement of PBR is that conjoint probability distributions (CPD) violate the Born rule, (CPD→~TBR). That means CPD make the quantum state unknown and hence epistemological. CPD mean the quantum state is not fixed by a determination of the HV. A given HV will produce more than one quantum state possibility—hence the quantum state is epistemological. Since ~CPD is the same as DPD and CPD implies a negation of the Born rule, CPD→~TBR, reversing the logic we get TBR→~CPD so TBR→DPD.

Let me add a few more comments. I believe that until the ontology/epistemology issue is fully resolved (although readers may believe it already resolved after reading Part I in this article), we still have the "measurement problem" that stimulated such considerations as given by PBR, Bell, Bohm, and many others. We also still have the nonlocality issue to deal with. Perhaps PBR can resolve this issue. Ontologically speaking, what does it mean to have nonlocal influences? What does it mean to have an observer effect (collapse of the QWF)? Does the PBR solution resolve these problems?

Consider the effect of observation on an ontic QWF. Does a human being alter the QWF simply by making an observation? If the QWF is ontic then we have a real observer effect—observation (including nonlocal) indeed alters the QWF and therefore reality. That would mean that mind is inextricably tied into matter; they are truly entangled and such a finding could lead to breaking discoveries in the study of consciousness. On the other hand, if the QWF proves to be epistemic (as defined by PBR) in violation of the Born probability rule, observation is simply the usage of the Bayesian approach to probabilities wherein new information simply changes what we know, but leaves reality unscathed—at least what we mean by ontic reality.

So let me summarize what we have garnered from PBR. Quantum wave functions are functions. That means they may depend on values of hidden variables to obtain values for themselves. Such hidden variables, like those found in thermodynamical functions, form probability distributions. Consequently if quantum wave functions

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http://arxiv.org/abs/1203.4779. Consequently Leifer doesn't think it is possible to establish that the QWF is epistemic purely by experiment. I wish to thank Matthew Pusey and particularly Matt Leifer for many helpful comments concerning quantum physics and epistemology.

must be constructed from such distributions of hidden variables, these distributions never can have overlaps. Overlapping distributions indicate lack of knowledge of the quantum wave function and therefore represent an epistemological situation which as a consequence leads to a violation of the Born rule of probability conservation. In brief to make quantum wave functions be at least as real (and therefore ontological) as say pressure is in thermodynamics their HV distributions must never overlap.

In the everyday world of our observations non-overlapping of probabilities means disjointed distributions, as seen, e.g., in Fig. 4, that lead to a world of objects behaving the laws of classical physics. In this next part of this article we shall consider another aspect of the epistemological/ontological question. Here the results of Bernhard Kochen & Ernst Paul Specker[19] (KS) will play a gigantic role. An inequality based on the KS theorem formed by four physicists: Alexander A. Klyachko, M. Ali Can, Sinem Binicioğlu, and Alexander S. Shumovsky (KCBS) will be discussed next and will contradict the results of PBR by showing that the assumption of PBR that HVs must lie within disjointed probability distributions cannot reproduce actual measurements made on quantum physical systems having single quantum wave functions.

PART II: HIDDEN VARIABLES: CONTEXTUALITY AND NON-CLASSICALITY

From what we have observed in Part I, given the validity of quantum physics, QWFs are to taken as real "out there" stuff if they can be based upon individual distributions of hidden variables that are not overlapping (disjointed). If on the other hand their hidden variable distributions overlap then the quantum wave functions will not reproduce the Born rule which merely takes into account the various probabilities predicted for various outcomes according to quantum physics. This violation of the Born rule would imply that quantum wave functions are not ontological but must be epistemological. In brief the violation would indicate quantum wave functions are not real and "out there" but merely a calculation tool.

So it would seem that reality of QWFs depends on how they represent the values of objects—their variables—hidden or not. In classical physics we encounter something similar in the field of thermodynamics. For example, we picture an ideal gas of N particles with each particle having a position, x, and a momentum, mv. We never actually observe these hidden variables but take it that the pressure, P, of such an enclosed gas in a volume, V, is given by $N\langle mv^2\rangle/3V$, where $\langle mv^2\rangle$ denotes twice the

[19] Kochen, S. & Specker, E. P. "The Problem of Hidden Variables in Quantum Mechanics." *J. Math. Mech.* **17,** 59 (1967)

average (expected) kinetic energy of a particle. Hence by calculating PV times 3/2 and dividing by the number of particles in the gas, N, we can determine the average kinetic energy of each particle. Our knowledge of each actual particle's kinetic energy is hidden from us, so we take that knowledge to be epistemological. Even so we take it that each particle really has a kinetic energy. We call such considerations the realm of classical physics.

Classicality: an example

Classicality also implies something more about hidden variables. Let me now look at a simple example one that will have far reaching implications for non-classicality. Take a coin. Examine carefully to see that it has two different sides heads (H) and tails (T). Suppose we assign a numerical value to each observed side say $a = +1$ for H and $a = -1$ for T. Now if we flip the coin, let it land, and observe one side; we can take our value assignment, say $+1$, for our H observation to be real and that the other unobserved side's value, -1, also to be real, unobserved, but nevertheless still "out there." We, in our everyday observations, take the world to consist of such objects and assign values to our observations even though we usually never actually see or count them all.

Now suppose we have five such coins. Further suppose we never actually see all the distributed values of the coins—that is they are hidden variables. We further suppose the coins are each contained in separate sectors of the apparatus shown in Fig. 9, so that upon performing a measurement we only see two adjoining coins. Even so, counting on the reality of the coins, it is not difficult to see that after flipping and landing, there are 32 (2^5) possible ways these five coins can show a side. If we assign a label, A_k, to designate the observation of the indicated values, $a_k = \pm 1$, associated with coin k and then multiple consecutive values, $a_k\, a_{k+1}$, corresponding (as shown in Fig. 10) to the observed pair of adjacent coins we can compute the following sum, S_j, expressed as a function of the values , a_k, $k \in (1, 2, 3, 4, 5)$ for a given run j (a run means flipping all five coins at once):

$$S_j\,(a_1, a_2, a_3, a_4, a_5) = \Sigma_k\, a_k\, a_{k+1} = a_1\, a_2 + a_2\, a_3 + a_3\, a_4 + a_4\, a_5 + a_5\, a_1,$$

$$(k \text{ modulo } 5) \qquad \text{eqn. 02.}$$

Only two coins are visible after flipping all five coins simultaneously. Which two can be viewed depends on the rotating pie sector that randomly can click into one of five possible positions obscuring three of the coins and exposing any two adjacent coins. We take it that the rotating pie sector is rotated after each flip before any observation is carried out.

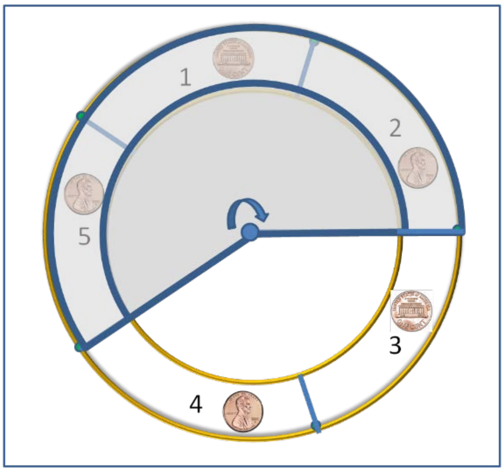

Fig. 10. Non-Contextuality measurements of two coins out of five. The pie sector (in gray) can rotate and click into positions where only two adjacent coins are visible at a time. Here coins 3 and 4 are observed while the remaining coins are hidden from view.

What is the lowest value S_j can take? Clearly the lowest value for S_j would occur when each term of eqn. 02 has the value −1. This could be arranged by having each a_k of the first four terms with alternate values, but then regardless of the sign of a_1 the fifth term must have value +1. Taking all possible ways these values can be assigned such that four terms each have the value −1, one remaining term must always have value +1; it is not difficult to see that in every such hidden alternate a_k values assignment, $S_j = -3$.

After a single flipping of all five coins, let us examine a typical pair of terms, such as $a_1 a_2$ and $a_2 a_3$. The question we ask is: can the value of a_2 depend on which of the two terms it is a part of? Commonsense tells us that the answer is no—whatever happens to coin 1 or coin 3 cannot influence what happens to coin 2. Mathematically we say the probabilities for all such terms consist of joint probabilities and therefore independent probabilities. We label this as a *non-contextual* situation—all such situations wherein measurements of a system's properties (here A_k) are able to be defined independently of both their own measurements and the measurements of any other systems (say $A_{k\pm1}$) define what is meant by non-contextuality.

In Table 1 I have shown just how S_j is computed for five separately weighted coins for each run, j. I have arbitrarily assigned consistent probabilities for heads (tails) to appear for each coin separately: for $a_1=\pm1$, $p_1=0.1$, 0.9; $a_2=\pm1$, $p_2=0.9$, 0.1; $a_3=\pm1$,

p_3=0.1, 0.9; a_4=±1, p_4=0.9, 0.1; and for a_5=±1, p_5=0.1, 0.9 where the plus sign indicates the 1st value for p, and the minus sign indicates the 2nd. Thus the probability, P_j, (where j indicates the number for a particular run) for each five-coin toss distribution of H and T is the product $p_1 p_2 p_3 p_4 p_5$.

j	a_1	a_2	a_3	a_4	a_5	$a_1 a_2$	$a_2 a_3$	$a_3 a_4$	$a_4 a_5$	$a_5 a_1$	S_j	P_j	$S_j P_j$	$P_j a_1 a_2$	$P_j a_2 a_3$	$P_j a_3 a_4$	$P_j a_4 a_5$	$P_j a_5 a_1$
1	1	1	1	1	1	1	1	1	1	1	5	0.00729	0.03645	0.00729	0.00729	0.00729	0.00729	0.00729
2	1	1	1	1	-1	1	1	1	-1	-1	1	0.00081	0.00081	0.00081	0.00081	0.00081	-0.00081	-0.00081
3	1	1	1	-1	1	1	1	-1	-1	1	1	0.00081	0.00081	0.00081	0.00081	-0.00081	-0.00081	0.00081
4	1	1	1	-1	-1	1	1	-1	1	-1	1	0.00009	0.00009	0.00009	0.00009	-0.00009	0.00009	-0.00009
5	1	1	-1	1	1	1	-1	-1	1	1	1	0.06561	0.06561	0.06561	-0.06561	-0.06561	0.06561	0.06561
6	1	1	-1	1	-1	1	-1	-1	-1	-1	-3	0.00729	-0.02187	0.00729	-0.00729	-0.00729	-0.00729	-0.00729
7	1	1	-1	-1	1	1	-1	1	-1	1	1	0.00729	0.00729	0.00729	-0.00729	0.00729	-0.00729	0.00729
8	1	1	-1	-1	-1	1	-1	1	1	-1	1	0.00081	0.00081	0.00081	-0.00081	0.00081	0.00081	-0.00081
9	1	-1	1	1	1	-1	-1	1	1	1	1	0.00081	0.00081	-0.00081	-0.00081	0.00081	0.00081	0.00081
10	1	-1	1	1	-1	-1	-1	1	-1	-1	-3	0.00009	-0.00027	-0.00009	-0.00009	0.00009	-0.00009	-0.00009
11	1	-1	1	-1	1	-1	-1	-1	-1	1	-3	0.00009	-0.00027	-0.00009	-0.00009	-0.00009	-0.00009	0.00009
12	1	-1	1	-1	-1	-1	-1	-1	1	-1	-3	0.00001	-0.00003	-0.00001	-0.00001	-0.00001	0.00001	-0.00001
13	1	-1	-1	1	1	-1	1	-1	1	1	1	0.00729	0.00729	-0.00729	0.00729	-0.00729	0.00729	0.00729
14	1	-1	-1	1	-1	-1	1	-1	-1	-1	-3	0.00081	-0.00243	-0.00081	0.00081	-0.00081	-0.00081	-0.00081
15	1	-1	-1	-1	1	-1	1	1	-1	1	1	0.00081	0.00081	-0.00081	0.00081	0.00081	-0.00081	0.00081
16	1	-1	-1	-1	-1	-1	1	1	1	-1	1	0.00009	0.00009	-0.00009	0.00009	0.00009	0.00009	-0.00009
17	-1	1	1	1	1	-1	1	1	1	-1	1	0.06561	0.06561	-0.06561	0.06561	0.06561	0.06561	-0.06561
18	-1	1	1	1	-1	-1	1	1	-1	1	1	0.00729	0.00729	-0.00729	0.00729	0.00729	-0.00729	0.00729
19	-1	1	1	-1	1	-1	1	-1	-1	-1	-3	0.00729	-0.02187	-0.00729	0.00729	-0.00729	-0.00729	-0.00729
20	-1	1	1	-1	-1	-1	1	-1	1	1	1	0.00081	0.00081	-0.00081	0.00081	-0.00081	0.00081	0.00081
21	-1	1	-1	1	1	-1	-1	-1	1	-1	-3	0.59049	-1.77147	-0.59049	-0.59049	-0.59049	0.59049	-0.59049
22	-1	1	-1	1	-1	-1	-1	-1	-1	1	-3	0.06561	-0.19683	-0.06561	-0.06561	-0.06561	-0.06561	0.06561
23	-1	1	-1	-1	1	-1	-1	1	-1	-1	-3	0.06561	-0.19683	-0.06561	-0.06561	0.06561	-0.06561	-0.06561
24	-1	1	-1	-1	-1	-1	-1	1	1	1	1	0.00729	0.00729	-0.00729	-0.00729	0.00729	0.00729	0.00729
25	-1	-1	1	1	1	1	-1	1	1	-1	1	0.00729	0.00729	0.00729	-0.00729	0.00729	0.00729	-0.00729
26	-1	-1	1	1	-1	1	-1	1	-1	1	1	0.00081	0.00081	0.00081	-0.00081	0.00081	-0.00081	0.00081
27	-1	-1	1	-1	1	1	-1	-1	-1	-1	-3	0.00081	-0.00243	0.00081	-0.00081	-0.00081	-0.00081	-0.00081
28	-1	-1	1	-1	-1	1	-1	-1	1	1	1	0.00009	0.00009	0.00009	-0.00009	-0.00009	0.00009	0.00009
29	-1	-1	-1	1	1	1	1	-1	1	-1	1	0.06561	0.06561	0.06561	0.06561	-0.06561	0.06561	-0.06561
30	-1	-1	-1	1	-1	1	1	-1	-1	1	1	0.00729	0.00729	0.00729	0.00729	-0.00729	-0.00729	0.00729
31	-1	-1	-1	-1	1	1	1	1	-1	-1	1	0.00729	0.00729	0.00729	0.00729	0.00729	-0.00729	-0.00729
32	-1	-1	-1	-1	-1	1	1	1	1	1	5	0.00081	0.00405	0.00081	0.00081	0.00081	0.00081	0.00081
sums →											0	1	-1.92	-0.64	-0.64	-0.64	0.64	-0.64

Table 1. Non-Contextuality Table for 5 unequally weighted coins.

As is easy to see by perusing the table, the minimum value for S_j (see entry for j=6, e.g.) is indeed −3. We can also compute S_j for each possible toss of the five coins multiplied by the probability, P_j, for that particular set of values: $S_j P_j$. We also find that $S_j P_j$ cannot be less than −3. If we add up the results found in the column labeled $S_j P_j$ we get the average or expectation value of these $A_k A_{k+1}$ measurements, where the brackets, "<>," denote expectation value:

$$<S_j (A_1, A_2, A_3, A_4, A_5)> = \Sigma_j P_j S_j (a_1, a_2, a_3, a_4, a_5)$$

$$= \Sigma_j P_j a_1 a_2 + \Sigma_j P_j a_2 a_3 + \Sigma_j P_j a_3 a_4 + \Sigma_j P_j a_4 a_5 + \Sigma_j P_j a_5 a_1. \qquad \text{eqn. 03.}$$

Since P_j is a probability, we note that $\Sigma_j P_j = 1$ as given at the bottom of the column labeled P_j. We also note as expressed in eqn. 03 that the average of the sums, $\Sigma_j P_j S_j$, is the sum of the averages for each term, $a_k\, a_{k+1}$,—something we expect to see in a linear equation and can be found in the table at the bottoms of the last five columns. Thus we find that as was the case for $S_j \geq -3$, for each j run, the same is true for $\Sigma_j P_j S_j \geq -3$, and for each and every $\Sigma_j a_k a_{k+1} P_j$: Hence:

$$\langle A_1 A_2\rangle + \langle A_2 A_3\rangle + \langle A_3 A_4\rangle + \langle A_4 A_5\rangle + \langle A_5 A_1\rangle \geq -3. \qquad \text{eqn. 04.}$$

Eqn. 04 is known as the Klyachko-Can-Binicioğlu-Shumovsky (KCBS) inequality[20] and must apply in any hidden variable theory where the probabilities for all such terms consist of disjoint probabilities and therefore independent probabilities. Note again that in eqn. 04 we write $\langle A_k A_{k+1}\rangle$ to denote the expectation (same as average) value not necessarily the quantum physical expectation value; that is,

$$\langle A_k A_{k+1}\rangle = \Sigma_j P_j a_k a_{k+1}. \qquad \text{eqn. 05.}$$

In the table we find for these so-weighted coins,

$$\langle S_j (A_1, A_2, A_3, A_4, A_5)\rangle = \Sigma_j P_j S_j = -1.92 \geq -3, \qquad \text{eqn. 06.}$$

in compliance with the KCBS inequality.

The important thing to note here is that the value taken by any measurement of A_k does not change in context when combined with neighboring $A_{k\pm1}$. And I repeat we call such observations non-contextual (e.g. the value obtained for A_3 is independent of A_2 or A_4.). So we take note that noncontextual hidden variable measurements are those in which the value of A_k is independent of whether we measure A_k together with A_{k-1} (which is compatible with A_k), or together with A_{k+1} (which is also compatible with A_k). A set of mutually compatible measurements is called a context.

It is possible that A_{k+1} and A_{k-1} may not be necessarily compatible. $\{A_k, A_{k-1}\}$ is one context and $\{A_k, A_{k+1}\}$ is a different one, and they may not both be contained in a joint context. Fig. 10 illustrates that coins within sections of the ring in non-adjacent positions are never measured at the same time—only adjacent sections are observed. Hence non-adjacent measurements are, in this sense, not compatible[21]—we can never measure them simultaneously in this set-up.

Yet noncontextuality applies here. The measurement of A_k will be the same in both contexts $\{A_k, A_{k-1}\}$ and $\{A_k, A_{k+1}\}$ regardless of which section it lies within. E. g., if, as in Fig. 9, the rotating pie sector covered sections 1, 2, and 3, exposing 4 and 5 instead

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[20] Klyachko, A., Can, M. A., Binicioğlu, S. & Shumovsky, A. S. "Simple Test for Hidden Variables in Spin-1 Systems." *Phys. Rev. Lett.* **101,** 020403 (2008).

[21] Here compatible means something rather simple—a covered coin and an uncovered coin are never observed simultaneously—hence they are incompatible.

of sections 5, 1, and 2, exposing 3 and 4, the coin showing in section 4 would not change value. Fig. 10, as clearly as I can make it, illustrates what is meant by noncontextuality in an non-compatible way, and even so, as a consequence, what is meant by classicality—the appearance of a classical world.

Classicality means non-contextuality

So why is contextuality import? Well it turns out that our everyday observations of things within our world of daily experience are non-contextual, whether compatible or not, and that is what we mean by a classically-perceived world. Such things are "out there" with both realized or observed values and hidden values. Nevertheless the hidden values are still taken to be "out there" and just as real as the values we do observe.[22] If two or more things are "out there" or even if a single thing has possible observable consequences such as a coin with two sides or a die with six, we infer the existence of those hidden values even though we don't actually observe them. Thus our observations are labeled classical. Classicality means we are logically consistent in applying values to possible, yet unobserved, observations as well as those values we do observe and assign such as the flight of a baseball shown in Fig. 1.

It also means that when we do make observations of the values of two or more things, these values are independent of each other provided the things are not connected in some manner. For example if we look again at the five coins example, whatever value we determine for say coin 3, is quite independent of the values that the other coins might give us. We say these values are non-contextual. This even if coins 1 and 3 are re-flipped, as long as we leave coin 2 alone, its value will not change. Examination of table 1 shows this as clearly as I can make it even if the coins are weighted unequally. But what would it mean if the value of a thing could change depending on the values of other things to which it has no connections? Such a world would mean classicality doesn't apply in the real world. It would also imply that quantum wave functions are not just simply "out there" in spite of the PBS result given above. Or perhaps better stated the observed consequences of quantum wave functions, meaning their observed eigenvalues when measurements are actually made, can change depending on how other measurements are made simultaneously in context with them.

[22] I should point out that we really don't observe values—we assign them based upon observations of things. Thus a meter may indicate a number from which I assign a value.

Non-Classicality means contextuality

When we consider measurements based within quantum physics, contextuality of observations comes into question and therefore so does classicality. The notion of contextuality probably first came into quantum physics in 1968. Two physicists, Simon Bernhard Kochen & Ernst Paul Specker (KS), came up with a rather perhaps complex but nevertheless surprising proof, a mathematical inequality, dealing with hidden variables;[23] specifically what we assume to be real and "out there," even if we don't actually look to see, turns out to be an illusion. KS concluded that quantum mechanical observables cannot represent "elements of physical reality." More specifically, they showed that any hidden variable theory that requires elements of physical reality to be non-contextual cannot be valid—i.e., will fail to predict observed results in some cases. Their theorem excludes hidden variable theories that are based on independence of the measurement arrangement—change the arrangement and the observations change. After KS's discovery of their inequality and how quantum physics violates it, a number of papers appeared attempting to simplify their theoretical results (I have listed some but not all[24]). Soon after a number of experimenters came on the scene and appeared to be carrying out experimental tests of contextuality based on our current interests in quantum computing.[25]

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[23] Kochen, S. & Specker, E. P. *Ibid.*

[24] Peres, A. "Two simple proofs of the Kochen-Specker theorem." *J. Phys. A: Math. Gen.* **24,** L175–L178 (1991).

Cabello, A., Estebaranz, J. M. & Garcia-Alcaine, G. "Bell-Kochen-Specker theorem: A proof with 18 vectors." *Phys. Lett. A* **212,** 183–187 (1996).

Pavicic, M., Merlet, J. P., McKay, B. & Megill, N. D. "Kochen–Specker vectors." *J. Phys. A: Math. Gen.* **38,** 1577–1592 (2005).

Toh, S. P. & Zainuddin, H. "Kochen–Specker theorem for a three-qubit system: A state-dependent proof with seventeen rays." *Phys. Lett. A* **374,** 4834–4837 (2010).

Cabello, A. "How many questions do you need to prove that unasked questions have no answers?" *Int. J. Quantum. Inform.* **4,** 55–61 (2006).

[25] Xi Kong, Mingjun Shi, Fazhan Shi, Pengfei Wang, Pu Huang, Qi Zhang, Chenyong Ju, Changkui Duan, Sixia Yu, and Jiangfeng Du. "An experimental test of the non-classicality of quantum mechanics using an unmovable and indivisible system." Source http://arxiv.org/abs/1210.0961v1.

Tao Li, Qiang Zeng, Xinbing Song & Xiangdong Zhang. "Experimental contextuality in classical light" www.nature.com/scientificreports/ *Scientific Reports* **volume7**, Article number: 44467 (2017).

Huang, Y.-F., Li, C.-F.; Zhang, Y.-S., Pan, J.-W. & Guo, G.-C. "Experimental Test of the Kochen-Specker Theorem with Single Photons." *Phys. Rev. Lett.* **90,** 250401 (2003).

Amselem, E., Radmark, M., Bourennane, M. & Cabello, A. "State-Independent Quantum Contextuality with Single Photons." *Phys. Rev. Lett.* **103,** 160405 (2009).

D'Ambrosio, V. *et al.* "Experimental Implementation of a Kochen-Specker Set of Quantum Tests." *Phys. Rev. X* **3,** 011012 (2013).

Hu, X.-M. *et al.* "Experimental Test of Compatibility-Loophole-Free Contextuality with Spatially Separated Entangled Qutrits". *Phys. Rev. Lett.* **117,** 170403 (2016).

In what follows I shall consider perhaps one of the simplest explanations of the type of the KS inequality: the KCBS inequality discussed above. KCBS looked at their inequality in terms of quantum physics to see if it still held. They considered a single 3-state spin-1 system often referred to as a "qutrit" whereas a qubit refers to a 2-state system such as the spin of an electron. The simplest example of a qutrit is a photon which can be polarized along any direction in space. Their question was what would happen to the KCBS inequality if measurements were taken with respect to a cyclic quintuplet of unit vectors, $\underline{d}_1$, $\underline{d}_2$, $\underline{d}_3$, $\underline{d}_4$, and $\underline{d}_5$ such that $\underline{d}_i \perp \underline{d}_{i+1}$, with the indices taken modulo 5 as shown in Figs. 11 and 12. Each unit vector radially stretches from the center of a unit sphere to five distinct points forming a pentagram lying on a circle of latitude of the sphere.

KCBS then looked at five different arrangements—each corresponding to a possible measurement of a qutrit's values when measured along three mutually perpendicular directions of space—and then considered five possible measurements. Each measurement performed simultaneously dealt with the qutrit's three possible states according to quantum physical laws. Here's how that worked.

Mazurek, M. D., Pusey, M. F., Kunjwal, R., Resch, K. J. & Spekkens, R. W. "An experimental test of noncontextuality without unphysical idealizations." *Nat. Commun.* **7,** 11780 (2016).
Bartosik, H. *et al.* "Experimental Test of Quantum Contextuality in Neutron Interferometry." *Phys. Rev. Lett.* **103,** 040403 (2009).
Hasegawa, Y., Loidl, R., Badurek, G., Baron, M. & Rauch, H. "Quantum Contextuality in a Single-Neutron Optical Experiment." *Phys. Rev. Lett.* **97,** 230401 (2006).
Kirchmair, G. *et al.* "State-independent experimental test of quantum contextuality." *Nature* **460,** 494–497 (2009).
Moussa, O., Ryan, C. A., Cory, D. G. & Laflamme, R. "Testing Contextuality on Quantum Ensembles with One Clean Qubit". *Phys. Rev. Lett.* **104,** 160501 (2010).
Dogra, S. & Dorai, K. Arvind. "Experimental demonstration of quantum contextuality on an NMR qutrit." *Phys. Lett. A* **380,** 1941–1946 (2006).
R. Lapkiewicz, P.Z. Li, C. Schaeff, N. K. Langford, S. Ramelow, M. Wiesniak, and A. Zeilinger, "Experimental non-classicality of a indivisible quantum system." *Nature* 474, 490 (2011).
Um, M. *et al.* "Experimental Certification of Random Numbers via Quantum Contextuality." *Sci. Rep.* **3,** 1627 (2013).

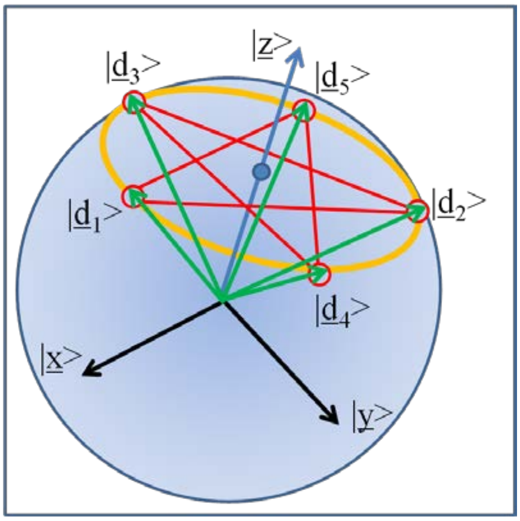

Fig. 11. The KCBS pentagram (in red, enclosed in the orange circle of latitude of the blue unit sphere) showing 5 unit vectors (in green) arranged so that any pair of radial unit vectors, di and di+1 are orthogonal. The blue vector labeled |z> = (0, 0, 1) pierces the pentagram at its center and it is the eigenvector corresponding to the 0 eigenvalue of the spin-1 projection along the z axis. Similarly |x> and |y> correspond to 0 eigenvalues of the spin-1 projections along the x and y axes resp.

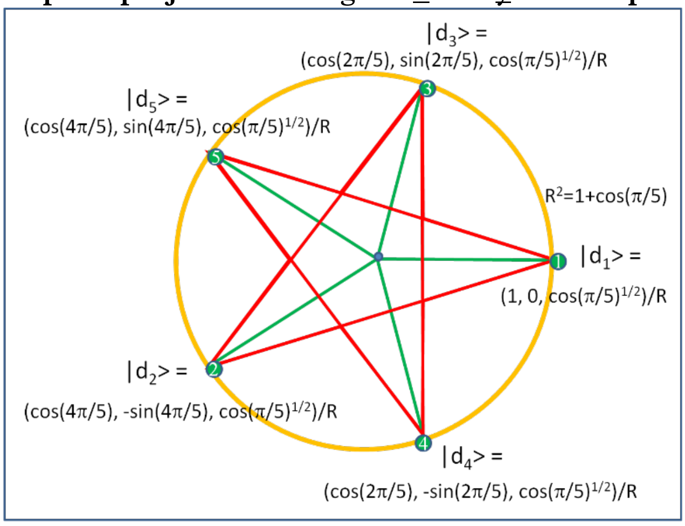

Fig. 12. Looking down on the KCBS pentagram (enclosed in the orange circle of latitude) from the north pole of its enclosing unit sphere. The unit vectors d1 through d5 are shown together with their component values.

Just as we illustrated above using five coins KCBS examined how a spin-1 system could be construed to yield five observables A_j, $j \in (1, 2, 3, 4, 5)$ taken in pairs $A_j A_{j+1}$, (j modulo 5. i.e., so that if j = 6 we roll j's value back to 1). In order to discuss the KCBS idea as clearly as I can, we will need to consider measurements of the spin or angular momentum of a spin-1 system. As such spin-1 is an observable of light commonly known as the polarization of a photon. A spin-1 system can be represented by its components or projections along three mutually perpendicular directions, say $\underline{x}$, $\underline{y}$, and $\underline{z}$, (where underlining means these are unit vectors along the three perpendicular axes of space). Now $\underline{x}$, $\underline{y}$, and $\underline{z}$, can be pointing in various directions so long as they remain mutually perpendicular.

In a diagonal representation,[26] the square of the spin component along the $\underline{x}$ direction can be written $J_{\underline{x}}^2 = I - |\underline{x}\rangle\langle\underline{x}|$, where "I" denotes the three dimensional identity matrix. Similarly we can write for the other mutually perpendicular projections: $J_{\underline{y}}^2 = I - |\underline{y}\rangle\langle\underline{y}|$, and $J_{\underline{z}}^2 = I - |\underline{z}\rangle\langle\underline{z}|$. In fact the projections of J^2 along any three mutually perpendicular directions will also be simultaneously measureable (these observable are said to commute). It is easy to show $J_{\underline{x}}^2$, $J_{\underline{y}}^2$, and $J_{\underline{z}}^2$ are mutually compatible and can be measured simultaneously. From these considerations it is possible to define observables, $A_{\underline{x}} = 2J_{\underline{x}}^2 - I = I - 2\,|\underline{x}\rangle\langle\underline{x}|$, and similarly for $A_{\underline{y}}$ and $A_{\underline{z}}$. It turns out that there are 3 eigenvectors and corresponding eigenvalues for each of these directions, $\underline{d}$, such that

$$A_{\underline{d}}\,|\underline{d}\rangle = -1\,|\underline{d}\rangle$$
$$A_{\underline{d}}\,|\underline{d}_{\perp}\rangle = +1\,|\underline{d}_{\perp}\rangle, \qquad \text{eqn. 07}$$

with the two directions perpendicular to $\underline{d}$ labeled $\underline{d}_{\perp}$. Hence we find in the complex Euclidean plane:[27]

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[26] The projection of the photon's spin or angular momentum along any direction in space can take on one of three possible values, ±1 and 0. The three mutually perpendicular directions of space are usually written, $\underline{x}$, $\underline{y}$, and $\underline{z}$. The underlining means these are unit vectors along their respective axes. As such it is possible to represent the spins of a photon as quantum physical operators J_x, J_y, and J_z. If we take $\underline{l}$ to be any one of these directions we can express $J_l\,\underline{\varphi} = i\,\underline{l} \otimes \underline{\varphi}$, where $\underline{\varphi}$ is a quantum state vector (a vector in complex Euclidean space) and "⊗" means vector cross product. Consequently, we find $\underline{\varphi} = \underline{l}$ to be a corresponding eigenvector of J_l with zero eigenvalue according to $J_l\,\underline{l} = 0$, and with $\underline{\varphi} = (\underline{m} \pm i\underline{n})/\sqrt{2}$, where $\underline{l}$, $\underline{m}$, $\underline{n}$ form any three mutually perpendicular unit vectors, $J_l\,(\underline{m} \pm i\underline{n})/\sqrt{2} = \pm 1\,(\underline{m} \pm i\underline{n})/\sqrt{2}$. We can also compute the squares of the spins along these directions. Further computation then leads to $J_l J_l\,\underline{\varphi} = J_l^2\,\underline{\varphi} = \underline{\varphi} - (\underline{l} \bullet \underline{\varphi})\,\underline{l}$. In operator language $J_l^2 = I - |\,\underline{l}\,\rangle\langle\,\underline{l}\,|$.

[27] See e.g., Jammer, Max. *The Philosophy of Quantum Mechanics*. NY: Wiley & Son. 1974. p. 324. We identify the Hilbert space of a spin-1 system with the complex Euclidean plane. As such, e.g.

$$A_{\underline{z}}\ |(\underline{x} \pm i\underline{y})/\sqrt{2}\rangle = +1\ |(\underline{x} \pm i\underline{y})/\sqrt{2}\rangle,$$
$$A_{\underline{z}}\ |\underline{z}\rangle = -1\ |\underline{z}\rangle. \qquad \text{eqns. 08.}$$

We find similar equations for $A_{\underline{y}}$ and $A_{\underline{z}}$. Suppose we now consider, as KCBS did, the five unit vectors labeled, $\underline{d}_1$, $\underline{d}_2$, $\underline{d}_3$, $\underline{d}_4$, and $\underline{d}_5$ such that these unit vectors form a pentagram as shown in Figs. 11 and 12. From the pentagram diagram, with $R^2 = 1 + \cos(\pi/5)$, where $\cos(\pi/5) = (1 + \sqrt{5})/4$, we compute the $\underline{x}$, $\underline{y}$, and $\underline{z}$ components of each unit $\underline{d}_i$ vector to be: [28]

$$\underline{d}_1 = (1, 0, \cos(\pi/5)^{1/2})/R,$$
$$\underline{d}_2 = (\cos(4\pi/5), -\sin(4\pi/5), \cos(\pi/5)^{1/2})/R,$$
$$\underline{d}_3 = (\cos(2\pi/5), \sin(2\pi/5), \cos(\pi/5)^{1/2})/R,$$
$$\underline{d}_4 = (\cos(2\pi/5), -\sin(2\pi/5), \cos(\pi/5)^{1/2})/R,$$
$$\underline{d}_5 = (\cos(4\pi/5), \sin(4\pi/5), \cos(\pi/5)^{1/2})/R. \qquad \text{eqns. 09.}$$

In terms of the $\underline{d}_i$ vectors we are looking at $A_{\underline{d}i} = I - 2\ |\underline{d}_i\rangle\langle\underline{d}_i|$ and the corresponding product $A_{\underline{d}i} A_{\underline{d}i+1} = I - 2\ |\underline{d}_i\rangle\langle\underline{d}_i|\ - 2\ |\underline{d}_{i+1}\rangle\langle\underline{d}_{i+1}|$, since $\langle\underline{d}_{i+1}|\underline{d}_i\rangle = 0$, because they are orthogonal. Let me now write A_i for $A_{\underline{d}i}$ so that $|\underline{d}_i\rangle$ is now written simply $|i\rangle$.

A quick look at $A_i A_{i+1}$ gives $I - 2\ |i\rangle\langle i|\ - 2\ |i + 1\rangle\langle i + 1|$. If we let $\underline{d}_\perp = i \otimes (i+1)$, where "$\otimes$" denote the vector cross product, then $\langle\underline{d}_\perp|i\rangle = \langle\underline{d}_\perp|i+1\rangle = 0$. Consequently we find for all i modulo 5,

$$A_i A_{i+1}\ |i \otimes (i + 1)\rangle = +1\ |i \otimes (i + 1)\rangle,$$
$$A_i A_{i+1}\ |\ i\rangle = -1\ |i\rangle,$$
$$A_i A_{i+1}\ |(i + 1)\rangle = -1\ |(i + 1)\rangle. \qquad \text{eqns. 10.}$$

KCBS then go on to consider the same inequality given in eqn. 04, but this time the brackets refer to computing the quantum physics expectation values of the $A_i A_{i+1}$ observables.

$$\langle A_1 A_2\rangle + \langle A_2 A_3\rangle + \langle A_3 A_4\rangle + \langle A_4 A_5\rangle + \langle A_5 A_1\rangle \geq -3,$$
$$\text{where } \langle A_i A_{i+1}\rangle = \langle\underline{\varphi}|\ A_i A_{i+1}\ |\underline{\varphi}\rangle. \qquad \text{eqn. 11.}$$

$J_z\,\underline{\varphi} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{\varphi}$. Hence $J_z\,(\underline{x} + i\underline{y})/\sqrt{2} = J_z \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}/\sqrt{2} = +1 \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}/\sqrt{2}$, or in terms of $J_z\,\underline{\varphi} = i\underline{z}\otimes\underline{\varphi}$, we have

$J_z\,(\underline{x} + i\underline{y})/\sqrt{2} = iz\otimes(\underline{x}+i\underline{y})/\sqrt{2} = (i\underline{y}+\underline{x})/\sqrt{2}$.

[28] With $r^2 = \cos(\pi/5) = -\cos(4\pi/5)$, $2r^4 - 1 = \cos(2\pi/5)$, and $R^2 = 1 + r^2$. Hence $\underline{d}_1 = (1, 0, r)/R$, $\underline{d}_2 = (-r^2, -(1-r^4)^{1/2}, r)/R$, $\underline{d}_3 = ((2r^4 - 1), 2r^2(1-r^4)^{1/2}, r)/R$, $\underline{d}_4 = ((2r^4 - 1), -2r^2(1-r^4)^{1/2}, r)/R$, $\underline{d}_5 = (-r^2, (1-r^4)^{1/2}, r)/R$.

So for each term we can always find a state vector $\underline{\varphi}$ such that $<\underline{\varphi}|\, A_i A_{i+1}\, |\underline{\varphi}> = -1$.[29] One might expect that the sum $\Sigma_i^5 <\underline{\varphi}|\, A_i A_{i+1}\, |\underline{\varphi}>$ (i modulo 5) would be able to reach -5 and thus violate the KCBS inequality. However this turns out to not be true in general. If we had chosen $\underline{\varphi}$ to be the eigenvector corresponding to $J_z^2\, |\underline{\varphi}_{\pm 1}> = +1|\underline{\varphi}_{\pm 1}>$, with $|\underline{\varphi}_{\pm 1}> = (|\underline{x}> \pm i\, |\underline{y}>)/\sqrt{2} = (1, \pm i, 0)/\sqrt{2}$ and we would find[30]

$$\Sigma_i^5 <\underline{\varphi}_{\pm 1}|\, A_i A_{i+1}\, |\underline{\varphi}_{\pm 1}> = -5 + 2\sqrt{5} = -0.5279 > -3, \qquad \text{eqn. 12.}$$

well within the classical range shown in Table 1, and so still in compliance with the KCBS inequality.

However KCBS did show how such a violation would occur when the expectation value, $\Sigma_i^5 <\underline{\varphi}|\, A_i A_{i+1}\, |\underline{\varphi}>$, is computed for $|\underline{\varphi}_0> = |\underline{z}> = (0, 0, 1)$, where $|\underline{\varphi}_0>$ is the "0" eigenfunction of J_z^2 (i.e., $J_z^2\, |\underline{z}> = 0$). In this case we easily find,[31]

$$\Sigma_i^5 <\underline{\varphi}_0|\, A_i A_{i+1}\, |\underline{\varphi}_0> = 5 - 4\sqrt{5} = -3.944 < -3, \qquad \text{eqn. 13.}$$

thus violating the KCBS inequality. KCBS point out that their violation of the KCBS inequality when (0, 0, 1) is used means their result is state-dependent as we can see by comparing eqns. 12 and 13. It also indicates that classical HVs cannot underlie QWFs even for a single quantum wave function as per a spin-1 system.

DISCUSSION OF PART II: DISJOINT AND CONJOINT HVS DO NOT EXIST

So we must conclude that quantum physics for certain states yields predictions that conform to experiments that do not conform to our classical intuitive notion that what we observe as real does not depend on what else we observe along with it. Contrarily change the "what else" and the observation itself changes. KCBS have shown this to be the case for a single photon hence consideration of nonlocality and entanglement does not enter their proof nor do I discuss this attribute of two or more particle systems. Since the KCBS paper, many experiments have been performed (see references in end note 25) confirming that indeed observations can and do change depending on what else is observed along with them.

We might have expected this result since the uncertainty principle tells us that we cannot observe certain pairs of observables such as momentum and position of an

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[29] For example either $<i|\, A_i A_{i+1}\, |\, i> = -1$ or , $<i+1|\, A_i A_{i+1}\, |\, i+1> = -1$.

[30] $J_{\underline{z}}\, |\underline{\varphi}_0> = 0$, and with $|\underline{\varphi}_{\pm 1}> = (|\underline{x}> \pm i|\underline{y}>)/\sqrt{2}$, $J_{\underline{z}}\, |\underline{\varphi}_{\pm 1}> = \pm 1\, |\underline{\varphi}_{\pm 1}>$.

[31] To see this consider that $\Sigma_i^5 <\underline{z}|\, A_i A_{i+1}\, |\underline{z}> = \Sigma_i^5 <\underline{z}|\, I\, |\underline{z}> - 4\, \Sigma_i^5 <\underline{z}|\, i><i\, |\underline{z}>$ where $<\underline{z}|\, i> = (0, 0, 1)\bullet(\underline{d}_{ix}, \underline{d}_{iy}, \underline{d}_{iz}) = \underline{d}_{iz} = r/R = 1/(5)^{1/4}$. Hence $\Sigma_i^5 <\underline{z}|\, i><i\, |\underline{z}> = 5/\sqrt{5} = \sqrt{5}$ and thus $\Sigma_i^5 <\underline{z}|\, A_i A_{i+1}\, |\underline{z}> = 5 - 4\sqrt{5}$.

object simultaneously. But the contextuality considerations of KCBS and others goes much farther than that, for they indeed consider simultaneous observation of commuting observables—those that can be observed simultaneously without changing values upon observation. In spite of the commutability of all observables considered, observations do not follow as our classically intuitive consideration would dictate—they are dependent on what other observations are made with them. What we see may not at all be what is "really out there," but instead may be dependent on what else has been observed, even if not by us.

As Lapkiewicz *et al* put it:[32] "Such incompatible properties, however, contrast strongly with what we experience in our everyday lives. If we look at a globe of the world, we can only see one hemisphere at any given time, but we suppose that the shapes of the continents on the far side remain the same irrespective of the observer's vantage point. Thus, by spinning the globe around to view different continents, we are able to construct a meaningful picture of the whole. It is reasonable to assume that observation reveals features of the continents that are present independent of which other continent we might be looking at. In this way, classical physics allows us to assign properties to a system without actually measuring it. All these properties can be assumed to exist in a consistent way, whether they are measured or not." Yet the experimental results of Lapkiewicz *et al* show that in spite of the obvious non-contextual appearance of globes and whatnot, fundamentally —at the quantum physics level— classical physics is wrong. Appearances of things depend on the contexts in which those things are observed.

As Cabello put it:[33] "Quantum correlations are contextual in the sense that they cannot be explained assuming that the result of a test A is independent of whether A is performed together with a compatible test B or with a compatible test C (which may be incompatible with B). This is the assumption of noncontextuality (NC) of results, and NC HV theories are those making this assumption. Two tests are compatible when, for any preparation, each test always yields an identical result, no matter how many times the tests are performed or in which order."

CONCLUSION

Is there a hidden nature of reality? Such a question arises naturally in quantum physics.

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[32] Quoted from: R. Lapkiewicz, P. Z. Li, C. Schaeff, N. K. Langford, S. Ramelow, M. Wiesniak, and A. Zeilinger, "Experimental non-classicality of a indivisible quantum system." *Nature* 474, 490 (2011).

[33] Quoted from: Cabello, A. "Simple Explanation of the Quantum Violation of a Fundamental Inequality." *Phys. Rev. Lett.* 110, 060402 (2013).

According to conclusions reached in Part I of this study there may be hidden variables which if known would yield the results we observe when measurements are carried out. However these hidden variables must exist "classically"—there cannot be situations in which a value for a HV can indicate two or more quantum wave functions simultaneously. Thus if an HV for a quantum wave function. φ_1, lies within a range of values ρ_1, and another HV for a quantum wave function. φ_2, lies within a range of values ρ_2, these ranges must not have overlapping values—they may be disjoint but not conjoint distributions. Hence from Part I we would conclude that if classical (ontological) HVs indeed exist then the necessary result that QWFs must also be ontological–really "out there" follows. But what if there is no such thing as classical real HVs?

On the other hand, in Part II the conclusion was there could not be classical hidden variables underlying reality according to expectations and measurements made based on KCBS's pentagram inequality. Quantum physical expectation values and measurements violate classicality—that is, any such classical HV theory in quantum physics does not conform to observations of the real world.

So we have found that the issue of the role of HV in questions of ontology and epistemology of states is particularly important when considering quantum physics because, it has led to different theories about how reality is constructed. It also tells us even if we can observe attributes simultaneously; their values can change depending on contextuality. So it seems that quantum physics is telling us that reality is constructed contextually and ontological realism of HVs and QWFs appears to be illusionary. Given that quantum physics underlies classical physics, then even classical physics must also be an illusion, or perhaps better put epistemologically constructed. What you see "out there" depends on the context you put on your observations "in here." No wonder there are so many different interpretations of "reality," "facts," and "opinions." No wonder that we live in a world stuffed with prejudices and fears. Also no wonder that the world is also filled with hope and dreams and all kinds of beliefs.

Thus we conclude that if quantum physics theory applies to real-world observations, the world cannot be a classical one—what we expect to see in it can and does depend ultimately on what context one makes in conjunction with one's observations as well as one's expectations. I believe this adds credence to the notion that it would be more fruitful to consider the "out there" as a product of the "in here"; in other words, quantum physics is telling us that the universe is a mental construction after all.[34]

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[34] Henry, Richard C. *ibid.* Also see: Kastrup, Bernardo. (2017) *ibid.* and Kastrup, B. (2014) *ibid.*

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